HEAT EQUATION–LINEAR AND NONLINEAR

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January 2024

- Definition: Euclidean norm $|x|^2 = \sum_{i=1}^{n} x_i^2$ in $\mathbb{R}^n$.
- Notation: $u_x = \frac{\partial u}{\partial x}, \quad u_t = \frac{\partial u}{\partial t}$.

Look for a function $u(x,t)$ satisfying

$$u_t - u_{xx} = 0, \quad x \in \mathbb{R},\ t \geq 0, \tag{1}$$

with given $u(x,0) = u_0(x)$.

THE HEAT KERNEL

Definition 1. *The function $G(x,t)$ given by*

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad x \in \mathbb{R},\ t > 0, \tag{2}$$

*is called the **heat kernel** (also: **the fundamental solution of the heat equation**).*

Claim 2. *The kernel $G(x,t) \in C^\infty(\mathbb{R} \times (0,\infty))$ and satisfies the heat equation there:*

$$G_t(x,t) - G_{xx}(x,t) = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty).$$

Claim 3. ***[positive summability kernel]*** *The kernel $G(x,t)$ is a positive summability kernel, namely, it has the following properties:*

- *(a)* $G(x,t) > 0, \quad (x,t) \in \mathbb{R} \times (0,\infty)$.
- *(b)* $\int_{\mathbb{R}} G(x,t)dx = 1, \quad t > 0$.
- *(c) For any* $\delta > 0$,

$$\lim_{t \downarrow 0} \int_{|x|>\delta} G(x,t)dx = 0.$$

SOLUTION OF INITIAL-VALUE PROBLEM (CAUCHY PROBLEM)

Lemma 4. *Let $u_0(x) \in L^\infty(\mathbb{R})$ and consider*

Date: February 10, 2024.

$$u(x,t) = \int_{\mathbb{R}} G(x-y,t)u_0(y)dy, \quad x \in \mathbb{R},\ t > 0. \tag{3}$$

Then

- $u(x,t) \in C^\infty(\mathbb{R} \times (0,\infty))$ *is a classical solution of the heat equation:*

$$u_t - u_{xx} = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty).$$

- *Let* $u_0(x)$ *be continuous at some* $x_0 \in \mathbb{R}$. *Then*

$$\lim_{x \to x_0, t \downarrow 0} u(x,t) = u_0(x_0). \tag{4}$$

Proof. See [1, Theorem 1, Sec. 2.3]. □

THE MAXIMUM PRINCIPLE

Let $Q = (a,b) \times (0,T)$ be a bounded cylinder.

Definition 5. *Let* $u(x,t) \in C^2(Q) \cap C(\overline{Q})$. *We say that* u ***is a classical subsolution of the heat equation*** *in* Q *if*

$$u_t - u_{xx} \leq 0, \quad (x,t) \in Q. \tag{5}$$

Lemma 6. *Let* $u(x,t) \in C^2(Q) \cap C(\overline{Q})$ *be a classical subsolution. Let*

$$\partial' Q = [a,b] \times [t=0] \cup \{a,b\} \times [0,T] \subseteq \partial Q.$$

Then

$$\max_{\partial' Q} u(x,t) = \max_{\overline{Q}} u(x,t). \tag{6}$$

Proof.
- Assume first that $u_t - u_{xx} < 0$ in Q. Let $Q_\varepsilon = (a,b) \times (0, T-\varepsilon)$ for a small $\varepsilon > 0$. Let $(\overline{x}, \overline{t}) \in \overline{Q_\varepsilon}$ be such that $u(\overline{x}, \overline{t}) = \max_{\overline{Q_\varepsilon}} u(x,t)$.
- If $(\overline{x}, \overline{t}) \notin \partial' Q_\varepsilon$ then it is easy to see that $u_t(\overline{x}, \overline{t}) - u_{xx}(\overline{x}, \overline{t}) \geq 0$. Contradiction.
- Thus $(\overline{x}, \overline{t}) \in \partial' Q_\varepsilon \subseteq \partial' Q$.
- Letting $\varepsilon \to 0$ and using the continuity of u in $\overline{Q}$ we get (6).
- Now assume only (5). Introduce, for a small $\eta > 0$ the function $v(x,t) = u(x,t) - \eta t$. Clearly $v_t - v_{xx} < 0$ in Q. Thus

$$\max_{\partial' Q} v(x,t) = \max_{\overline{Q}} v(x,t).$$

- Take the limit $\eta \to 0$.

□

Corollary 7. *Let $u(x,t) \in C^2(Q) \cap C(\overline{Q})$ be a classical solution of the heat equation in Q.*

Assume that $u \equiv 0$ on $\partial' Q$. Then

$$u \equiv 0 \text{ in } Q.$$

MAXIMUM PRINCIPLE FOR THE CAUCHY PROBLEM

Theorem 8. *Let $u(x,t)$ be a classical solution of the heat equation in $\mathbb{R} \times (0,T)$ such that $u \in C_b(\mathbb{R} \times [0,T]) = L^\infty(\mathbb{R} \times [0,T]) \cap C(\mathbb{R} \times [0,T])$. Then*

$$u(x,t) \le \sup_{\mathbb{R}} u_0(x), \quad (x,t) \in \mathbb{R} \times [0,T], \ u_0(x) = u(x,0). \tag{7}$$

In fact, it is sufficient to assume $u_t - u_{xx} \le 0$ in $\mathbb{R} \times (0,T)$.

Proof.

- For every $0 < \varepsilon < T$ the function
$$\Phi(x,y,t) = \frac{1}{\sqrt{4\pi(T+\varepsilon-t)}} e^{\frac{(x-y)^2}{4(T+\varepsilon-t)}}$$
satisfies
$$\Phi_t - \Phi_{xx} = 0, \quad (x,t) \in \mathbb{R} \times (0,T).$$
- For every small $\mu > 0$ the function $v_\mu(x,y,t) = u(x,t) - \mu\Phi(x,y,t)$ satisfies, for any fixed $y \in \mathbb{R}$,
$$\frac{\partial}{\partial t} v_\mu(x,y,t) - \frac{\partial^2}{\partial x^2} v_\mu(x,y,t) \le 0, \quad (x,t) \in \mathbb{R} \times (0,T).$$
- Fix $y \in \mathbb{R}$ and $\rho > 0$ and define
$$Q_\rho = (y-\rho, y+\rho) \times (0,T).$$
From the maximum principle for a bounded domain we have
$$\max_{(x,t) \in \overline{Q_\rho}} v_\mu(x,y,t) \le \max_{(x,t) \in \partial' Q_\rho} v_\mu(x,y,t).$$
- Clearly
$$\lim_{\rho \to \infty} \max_{(x=y\pm\rho, t \in [0,T])} v_\mu(x,y,t) = 0,$$
so taking ρ sufficiently large
$$\sup_{(x,t) \in \mathbb{R} \times [0,T]} v_\mu(x,y,t) \le \sup_{x \in \mathbb{R}} v_\mu(x,y,0) \le \sup_{\mathbb{R}} u_0(x).$$
- Let $\mu \to 0$ to conclude the proof.

□

Now in analogy to Corollary 7

Corollary 9. *[uniqueness] Let $u(x,t) \in C^2(\mathbb{R} \times (0,T)) \cap C_b(\mathbb{R} \times [0,T])$ be a classical solution of the heat equation in $\mathbb{R} \times (0,T)$.*

Assume that $u_0(x) = u(x,0) = 0$ for all $x \in \mathbb{R}$. Then

$$u \equiv 0 \text{ in } \mathbb{R} \times [0,T].$$

INFINITE PROPAGATION SPEED

Claim 10. *Let the initial function $u_0(x)$ be nonnegative and compactly supported, but not identically zero. Let $u(x,t)$ be the classical solution to the heat equation.Then for all $(x,t) \in \mathbb{R} \times (0,\infty)$ we have $u(x,t) > 0$.*

NONHOMOGENEOUS HEAT EQUATION–DUHAMEL'S PRINCIPLE

We follow [1, Section 2.3].

We consider the nonhomogeneous Cauchy problem

$$\begin{cases} u_t - u_{xx} = f(x,t), \quad (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = 0, \quad x \in \mathbb{R}. \end{cases} \tag{8}$$

We assume that $f \in C_0^2(\mathbb{R} \times [0,\infty)$.

Theorem 11. ***[Duhamel's principle]*** *The unique solution of* (8) *is given by*

$$u(x,t) = \int_0^t \int_{\mathbb{R}} G(x-y,t-s) f(y,s) dy\, ds. \tag{9}$$

Proof.

- We change variables in the integration to express the right-hand side of (9) as

$$u(x,t) = \int_0^t \int_{\mathbb{R}} G(x-y,t-s) f(y,s) dy\, ds = \int_0^t \int_{\mathbb{R}} G(y,s) f(x-y,t-s) dy\, ds.$$

  Then

- $$u_t(x,t) = \int_{\mathbb{R}} G(y,t) f(x-y,0) dy + \int_0^t \int_{\mathbb{R}} G(y,s) f_t(x-y,t-s) dy\, ds, \ (x,t) \in \mathbb{R} \times (0,\infty).$$

  $$u_{xx}(x,t) = \int_0^t \int_{\mathbb{R}} G(y,s) f_{xx}(x-y,t-s) dy\, ds.$$

  Note that the derivatives are continuous in $\mathbb{R} \times (0,\infty)$.
- We want to integrate by parts and use the fact that G solves the heat equation. However we need to treat separately the singularity of $G(y,s)$ at $s = 0$. Thus, we take a small $\varepsilon > 0$ and write
- $$\begin{aligned} u_t(x,t) - u_{xx}(x,t) &= \int_{\mathbb{R}} G(y,t) f(x-y,0) dy \\ &+ \int_0^{\varepsilon} \int_{\mathbb{R}} G(y,s)[-f_s(x-y,t-s) - f_{yy}(x-y,t-s)] dy\, ds \\ &+ \int_{\varepsilon}^t \int_{\mathbb{R}} G(y,s)[-f_s(x-y,t-s) - f_{yy}(x-y,t-s)] dy\, ds \\ &= K + J_\varepsilon + I_\varepsilon. \end{aligned}$$

- The estimate for J_ε is straightforward:

$$|J_\varepsilon| \le C \int_0^{\varepsilon} \int_{\mathbb{R}} G(y,s) dy\, ds = C\varepsilon.$$

- In I_ε we can integrate by parts:

$$I_\varepsilon = \int_\varepsilon^t \int_{\mathbb{R}} [G_s(y,s) - G_{yy}(y,s)] f(x-y,t-s) dy\, ds$$
$$+ \int_{\mathbb{R}} G(y,\varepsilon) f(x-y,t-\varepsilon) dy - \int_{\mathbb{R}} G(y,t) f(x-y,0) dy$$
$$= \int_{\mathbb{R}} G(y,\varepsilon) f(x-y,t-\varepsilon) dy - K.$$

- So $I_\varepsilon + K = \int_{\mathbb{R}} G(y,\varepsilon) f(x-y,t-\varepsilon) dy$ and

$$u_t(x,t) - u_{xx}(x,t) = \lim_{\varepsilon\to 0}(I_\varepsilon + K) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}} G(y,\varepsilon) f(x-y,t-\varepsilon) dy = f(x-y,t)|_{y=0} = f(x,t).$$

□

A MORE GENERAL FORM OF DUHAMEL'S PRINCIPLE

Fix $T > 0$. Denote $Q_T = \mathbb{R} \times (0,T)$. Recall that

$$C_b(\overline{Q_T}) = C(\overline{Q_T}) \cap L^\infty(Q_T).$$

Theorem 12. *Let $f(x,t) \in C_b(\overline{Q_T})$ be globally Hölder in $x \in \mathbb{R}$, namely, for some constants $L > 0,\ 0 < \alpha \le 1$,*

$$|f(x_2,t) - f(x_1,t)| \le L|x_2 - x_1|^\alpha, \quad x_1,\, x_2 \in \mathbb{R},\ t \in [0,T]. \tag{10}$$

Then the function

$$u(x,t) = \int_0^t \int_{\mathbb{R}} G(x-y,t-s) f(y,s) dy\, ds, \tag{11}$$

satisfies:

- *$u(x,t) \in C_b(\overline{Q_T})$.*
- *$u(x,t)$ is a classical solution of*

$$\begin{cases} u_t - u_{xx} = f(x,t), & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = 0, & x \in \mathbb{R}. \end{cases} \tag{12}$$

Proof. Let $M = \|f\|_\infty = \sup_{(x,t)\in\overline{Q_T}} |f(x,t)|$. Then

$$\Big| \int_0^\delta \int_{\mathbb{R}} G(x-y,t-s) f(y,s) dy\, ds \Big| \le M\delta.$$

Thus, it suffices to take a small $h > 0$, study $u(x,t)$ for $t > h$ and finally let $h \to 0$.

We therefore define

$$u_h(x,t) = \int_0^{t-h} \int_{\mathbb{R}} G(x-y,t-s) f(y,s) dy\, ds.$$

- Clearly $u_h(x,t) \in C_b(\mathbb{R} \times [h,T])$ and, for any $0 < t_1 < t_2 \le T$,

$$\lim_{h\to 0} u_h(x,t) = u(x,t), \quad \text{uniformly in } \mathbb{R} \times [t_1,t_2].$$

We conclude that $u(x,t) \in C_b(\overline{Q_T})$.

- Now, for $t > h$,

$$\frac{\partial}{\partial t}u_h(x,t) = \int_{\mathbb{R}} G(x-y,h)f(y,t-h)dy + \int_0^{t-h}\int_{\mathbb{R}} \frac{\partial}{\partial t}G(x-y,t-s)f(y,s)dy\,ds,$$

and since $\int_{\mathbb{R}} \frac{\partial}{\partial t}G(x-y,t-s)dy = 0$,

$$\frac{\partial}{\partial t}u_h(x,t) = \int_{\mathbb{R}} G(x-y,h)f(y,t-h)dy + \int_0^{t-h}\int_{\mathbb{R}} \frac{\partial}{\partial t}G(x-y,t-s)(f(y,s)-f(x,s))dy\,ds = I_h(x,t) + J_h(x,t). \tag{13}$$

Furthermore, since $\frac{\partial}{\partial t}G(x,t) = \frac{\partial^2}{\partial x^2}G(x,t)$ we get

$$\frac{\partial^2}{\partial x^2}u_h(x,t) = J_h(x,t). \tag{14}$$

- **Evaluating the limit of $I_h(x,t)$.**
  We estimate

$$|I_h - f(x,t)| \le \Big|\int_{\mathbb{R}} G(x-y,h)(f(y,t-h)-f(x,t-h))dy\Big| + |f(x.t-h)-f(x,t)|$$
$$\le L\int_{\mathbb{R}} G(x-y,h)|x-y|^\alpha dy + |f(x,t-h)-f(x,t)|,$$

so clearly

$$\lim_{h\to 0} I_h(x,t) = f(x,t) \text{ uniformly for } (x,t) \text{ in compact subsets of } Q_T. \tag{15}$$

- **Proof of convergence of** $J_h(x,t)$. We compute

$$\frac{\partial}{\partial t}G(x,t) = (4\pi t)^{-\frac{1}{2}}\Big[\frac{x^2}{4t^2} - \frac{1}{2}t^{-1}\Big]e^{-\frac{x^2}{4t}}.$$

Using the Lipschitz condition (10) and taking $0 < h_1 < h_2$ we get

$$|J_{h_2} - J_{h_1}| \le L\int_{t-h_2}^{t-h_1}\int_{\mathbb{R}} (4\pi(t-s))^{-\frac{1}{2}}\Big|\frac{(x-y)^2}{4(t-s)^2} - \frac{1}{2}(t-s)^{-1}\Big|e^{-\frac{(x-y)^2}{4(t-s)}}|x-y|^\alpha dy\,ds.$$

We change y to a new variable z given by

$$z = \frac{x-y}{\sqrt{4(t-s)}}, \quad dz = -\frac{dy}{\sqrt{4(t-s)}}.$$

We get, with some constant $C > 0$,

$$|J_{h_2} - J_{h_1}| \le CL\int_{t-h_2}^{t-h_1}(t-s)^{-1+\frac{\alpha}{2}}ds \cdot \int_{\mathbb{R}}(|z|^\alpha + |z|^{2+\alpha})e^{-z^2}dz$$

hence

$$|J_{h_2} - J_{h_1}| \to 0 \text{ as } h_1,\, h_2 \to 0.$$

- In view of (13) and (14) we conclude that

The derivatives $\frac{\partial}{\partial t}u_h(x,t)$ and $\frac{\partial^2}{\partial x^2}u_h(x,t)$ converge, as $h \to 0$, for all $(x,t) \in Q_T$, uniformly in every compact subset.

- By the convergence of u_h and standard calculus facts we must have

$$\begin{cases} \lim\limits_{h\to 0} \frac{\partial}{\partial t} u_h(x,t) = \frac{\partial}{\partial t} u(x,t), \\ \lim\limits_{h\to 0} \frac{\partial^2}{\partial x^2} u_h(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \end{cases} \tag{16}$$

and the convergence is uniform in every compact subset of Q_T.
- Now $\frac{\partial}{\partial t} u_h(x,t) = I_h + J_h$ and $\frac{\partial^2}{\partial x^2} u_h(x,t) = J_h$, so

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) &= \lim_{h\to 0} \Big[\frac{\partial}{\partial t} u_h(x,t) - \frac{\partial^2}{\partial x^2} u_h(x,t) \Big] \\ &= \lim_{h\to 0} I_h(x,t) = f(x,t), \quad (x,t) \in Q_T. \end{aligned}$$

□

NONLINEAR HEAT EQUATION

(We follow [1, Section 4.4]).

We consider the nonlinear equation

$$\begin{cases} v_t - \varepsilon v_{xx} + (v_x)^2 = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty),\ \varepsilon > 0, \\ v(x,0) = v_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{17}$$

Remark 13. *The equation* (17) *is sometimes called the* ***"viscous Hamilton-Jacobi equation".***

Assume that v is a classical solution and define a new function $w(x,t)$ by

$$w(x,t) = e^{-\frac{v(x,t)}{\varepsilon}}. \tag{18}$$

Claim 14. *The function w satisfies the linear heat equation:*

$$\begin{cases} w_t - \varepsilon w_{xx} = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty), \\ w(x,0) = e^{-\frac{v_0(x)}{\varepsilon}}, \quad x \in \mathbb{R}. \end{cases} \tag{19}$$

Definition 15. *The transformation* (19) *is called the* ***Cole-Hopf transformation.***

THE VISCOUS BURGERS EQUATION

We consider the Burgers equation with "viscosity", namely:

$$\begin{cases} u_t - \varepsilon u_{xx} + 2uu_x = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{20}$$

Set

$$v(x,t) := \int_{-\infty}^{x} u(y,t)dy,$$

then

$$v_{xt} - \varepsilon v_{xxx} + [v_x^2]_x = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty).$$

If $v(x,t)$ and its derivatives decay as $x \to -\infty$, we can integrate this equation to get

$$\begin{aligned} v_t - \varepsilon v_{xx} + v_x^2 = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty), \\ v(x,0) = \int_{-\infty}^{x} u_0(y)dy. \end{aligned} \tag{21}$$

This is exactly Equation (17).

References

[1] L. C. Evans, *"Partial Differential Equations"*, American Mathematical Society 1998.
[2] E. Godlewski and P.-A. Raviart, *"Hyperbolic Systems of Conservation Laws"*, Ellipses, 1991.

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